Frictionless dynamics of Bose-Einstein condensates under fast trap variations

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Abstract. A method is proposed to design the time dependence of the trap frequency and achieve in a short time an adiabatic-like (frictionless) evolution of Bose-Einstein condensates governed by the Gross-Pitaevskii equation. Different cases depending on the effective dimension of the trap and the interaction regimes are considered. 2D traps are particularly suitable as the method can be applied without the need to impose any additional time-dependent change in the strength of the interatomic interaction or a Thomas-Fermi regime as it occurs for 1D and 3D traps.

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1. Introduction

In order to manipulate Bose-Einstein condensates for different applications it is important to study and control their response to time-dependent changes of the confining fields. A natural approach to avoid undesired excitations is to modify the trap adiabatically, i.e., very slowly, so that, if the initial state is in the ground state the final state will be the ground state as well. However, this may require very long times and become impractical. Faster changes are thus a desirable objective but they will in general induce excitations and oscillations (inner frictional heating [1]), so that the proportion of the ground state in the final state may be small [2, 3, 4]. These difficulties raise the question addressed in this paper: Is it possible to change the trap in a very short time, taking the condensate, up to a global phase, to the same state that would be reached after a slow (adiabatic) process? This question has been answered recently in the affirmative for cooling expansions within the framework of the linear Schrödinger equation [5]. For preliminary work see [6, 7]. The method used to design the time-dependence of the trap frequency was based on Lewis-Riesenfeld invariants of motion [8] and simple inverse scattering techniques that had been applied for complex potential optimization [9]. Our objective here is to analyze if and how the same techniques used in that simple case can be adapted to non-linear interactions and systems described

by a Gross-Pitaevskii (GP) equation. As we shall see, the applicability of the method will depend critically on the effective dimension of the trap. We shall first discuss for simplicity with some detail one dimensional (1D) traps, and then 2D and 3D traps subjected to time-dependent frequencies. By 1D traps we mean quasi-1D cigar-shaped traps with tight (fixed) transversal confinement where the axial frequency is varied in time; similarly 2D traps are quasi-2D disk-shaped traps with tight, fixed, axial confinement in which the transversal frequency is varied; and finally, the 3D traps refer to harmonic traps with spherical symmetry. We assume in all cases that a GP equation can be derived corresponding to each dimensionality, and use g generically for the coupling parameter of the non-linear term even though it is different for the three cases [10].

2. One dimensional traps

Our starting point is the effective 1D Gross Pitaevski equation for the longitudinal (x) direction in an elongated cigar trap,

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega(t)^2x^2 + g|\psi|^2\right]\psi, \tag{1}$$

g being the coupling parameter. The application of the invariant concept here is not as simple as for the Schrödinger equation [11], so we shall use instead an approach which leads in that case to the same results. The idea is to assume for the wavefunction the ansatz [12]

$$\psi(x,t) = e^{-\beta(t)}e^{-\alpha(t)x^2}\phi(x,t). \tag{2}$$

Substituting this into Eq. (1), and using the scaling $\rho = x/b$ and redefined wavefunction $\Phi(\rho,t) = \phi(x,t)$, we get

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{b^2}\frac{\partial^2\Phi}{\partial\rho^2} + \left[\frac{1}{2}m\omega(t)^2 + i\hbar\dot{\alpha} - \frac{2\hbar^2}{m}\alpha^2\right]b^2\rho^2\Phi$$
$$+ \left[ge^{-(\alpha+\alpha^*)x^2}e^{-(\beta+\beta^*)}|\Phi|^2\right]\Phi + \left[i\hbar\dot{\beta} + \frac{\hbar^2\alpha}{m}\right]\Phi + \left[2\frac{\hbar\alpha}{m} + i\frac{\dot{b}}{b}\right]\hbar\rho\frac{\partial\Phi}{\partial\rho}, \tag{3}$$

where the dot means derivative with respect to time. Let us now impose that the coefficients in square brackets [...] of the last two terms vanish. This means that (we assume b real)

$$\beta = \frac{1}{2}\ln b, \quad \alpha = -\frac{im}{2\hbar}\frac{\dot{b}}{b}, \tag{4}$$

and $e^{-(\alpha+\alpha^*)x^2}e^{-(\beta+\beta^*)} = b^{-1}$. Suppose now that the coefficient of $b^2\rho^2\Phi$ in (3) is made constant, equal to $m\omega_0^2/(2b^4)$ (for an alternative see the final discussion), where $\omega_0 = \omega(0)$. Using (4) this is equivalent to imposing for b and $\omega(t)$ an Ermakov equation,

$$\ddot{b} + \omega(t)^2 b = \frac{\omega_0^2}{b^3}. \tag{5}$$

It is useful to express the resulting wave equation in terms of a new scaled time,

$$\tau(t) = \int_0^t \frac{dt'}{b^2}, \tag{6}$$

and wavefunction $\Psi(\rho, \tau) = \phi(\rho, t)$,

$$i\hbar \frac{\partial \Psi}{\partial \tau} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{m\omega_0^2}{2}\rho^2\Psi + gb|\Psi|^2\Psi. \tag{7}$$

For $g = 0$ this is the Schrödinger equation of a time-independent harmonic oscillator. The evolution of ψ has thus been conveniently mapped to the simple solution of an auxiliary stationary system. Choosing $b(0) = 1$, $\dot{b}(0) = 0$ the "auxiliary" (7) and physical (1) oscillators coincide at $t = 0$, so any instantaneous eigenstate of $t = 0$, with vibrational quantum number n and energy $E_n = \hbar\omega_0(n + 1/2)$, evolves according to a propagating mode determined by Eqs. (2,4) and the solution of the Ermakov equation $b(t)$,

$$\psi(x, t) = b^{-1/2} e^{\frac{im}{2\hbar}\frac{\dot{b}}{b}x^2} e^{-iE_n\tau(t)/\hbar}\Psi_n(x/b, 0). \tag{8}$$

In general this mode will not coincide with the instantaneous eigenstate of the physical Hamiltonian $H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega(t)^2x^2$, unless $b(t) = [\omega_0/\omega(t)]^{1/2}$ and $\dot{b}(t) = 0$, up to the global phase factor $e^{-iE_n\tau(t)/\hbar}$. This motivated our proposal in [5]: it is an inverse method in which, given the initial ω_0 and final frequencies $\omega_f = \omega(t_f)$, the intermediate trajectory $\omega(t)$ is left undetermined at first and the boundary conditions

$$b(0) = 1, \quad \dot{b}(0) = 0, \tag{9}$$

$$b(t_f) = (\omega_0/\omega_f)^{1/2}, \quad \dot{b}(t_f) = 0 \tag{10}$$

are imposed at initial and final times $t = 0, t_f$ (they also imply a vanishing $\ddot{b}$ at these two times to satisfy Eq. (5)). $b(t)$ is then interpolated with some functional form, e.g., a polynomial with enough coefficients to satisfy all conditions, and finally $\omega^2(t)$ is calculated from the Ermakov equation (5). This generates, in particular, very fast phase-space conserving cooling processes where $\omega^2(t)$ takes during some time interval negative values, i.e., the trap becomes an expulsive potential.

If $g \neq 0$ the coefficient of the non-linear term in the auxiliary equation is generally time dependent. Thus, imposing $\dot{b}(t_f) = 0$ eliminates the phase-factor $e^{-\alpha(t_f)x^2}$ but nothing guarantees that $\Psi(\tau(t_f))$ is proportional to the instantaneous eigenstate of the GP equation at t_f. A way out, in principle, is to make the coupling coefficient time-dependent with the aid of a Feshbach resonance as $g(t) = g_0/b(t)$, with g_0 constant. The resulting auxiliary equation has then time-independent coefficients,

$$i\hbar \frac{\partial \Psi}{\partial \tau} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{m\omega_0^2}{2}\rho^2\Psi + g_0|\Psi|^2\Psi. \tag{11}$$

and can be solved in the form $e^{-i\mu\tau(t)/\hbar}\Psi(x/b, 0)$, where μ is the chemical potential for the initial trap, so that

$$\psi(x, t) = b^{-1/2} e^{i\frac{im}{2\hbar}\frac{\dot{b}}{b}x^2} e^{-i\mu\tau(t)/\hbar}\Psi(x/b, 0), \tag{12}$$

and the same inverse method described for the Schrödinger equation can now be applied to design a fast frictionless process for the ground state condensate. One can easily check that, keeping $b(t) = b_f$ constant for $t > t_f$, which results in $\omega(t) = \omega_f$ and $g = g_0(\omega_f/\omega_0)^{1/2}$ for $t > t_f$, the solution $\psi(x,t)$ of (1) given by (12) becomes stationary, with a new scaled chemical potential $\mu/b(t_f)^2$.

Other special case is a "Thomas Fermi" (TF) limit, keeping g constant. Using a modified Ermakov equation and a different time scaling

$$\ddot{b} + \omega(t)^2 b = \frac{\omega_0^2}{b^2}, \quad \tau(t) = \int_0^t \frac{dt'}{b}, \tag{13}$$

render an auxiliary equation with time-independent coefficients for the non-linear and harmonic potential terms. If $g|\Psi|^2/(\hbar\omega_0) \gg 1$ the kinetic term may be neglected,

$$i\hbar\frac{\partial\Psi}{\partial\tau} = \frac{m\omega_0^2}{2}\rho^2\Psi + g|\Psi|^2\Psi. \tag{14}$$

This equation can also be solved by separation of variables, $\Psi(x/b,\tau) = e^{-i\mu\tau/\hbar}\Psi(x/b,0)$, and $\psi(x,t)$ takes again the form of Eq. (12), with different values for μ, τ, b, and the initial wavefunction. Note that this TF approximation is carried out in the auxiliary equation, and not at the level of the original GP equation, since that would imply a frozen density [12, 2]. From the modified Ermakov equation in (13), the inversion method to find a frictionless trajectory $\omega(t)$ requires in this 1D-TF scenario to change the boundary condition at t_f in (10) to $b(t_f) = (\omega_0/\omega_f)^{2/3}$, with $\ddot{b}(0) = \ddot{b}(t_f) = 0$ as before.

3. Two and three dimensional traps

The manipulations in 1D suggest for 2D and 3D a wavefunction ansatz of the form [2]

$$\psi(\mathbf{r},t) = b^{-d/2} e^{\frac{imr^2}{2\hbar}\frac{\dot{b}}{b}}\,\phi(\mathbf{r},t), \tag{15}$$

where d is the dimension, $r = (x^2+y^2)^{1/2}$ in 2D or $r = (x^2+y^2+z^2)^{1/2}$ in 3D. This form guarantees an auxiliary equation without first spatial derivatives.

With $\boldsymbol{\rho} = \mathbf{r}/b$ and a notation for the wavefunctions parallel to the 1D case there results, by substituting (15) into the 2D or 3D GP equations,

$$i\hbar\frac{\partial\Psi}{\partial\tau}\left(\frac{d\tau}{dt}b^2\right) = -\frac{\hbar^2}{2m}\Delta_\rho\Psi + \frac{m}{2}\left[\omega^2(t) + \frac{\ddot{b}}{b}\right]\rho^2 b^4\Psi + \frac{g}{b^{d-2}}|\Psi|^2\Psi, \tag{16}$$

where the Laplacian should be adapted to the dimension, $\Psi = \Psi(\boldsymbol{\rho},\tau)$, and τ has not been specified yet. (This equation includes the case $d = 1$ too by substituting $r \to x$ and the Laplacian by a second derivative.)

In 2D, the ordinary Ermakov equation (5) and the τ in Eq. (6) are the optimal choice since all coefficients in the auxiliary equation (assuming a constant g) become time independent, even outside the Thomas-Fermi regime,

$$i\hbar\frac{\partial\Psi}{\partial\tau} = -\frac{\hbar^2}{2m}\Delta_\rho\Psi + \frac{m\omega_0^2}{2}\rho^2\Psi + g|\Psi|^2\Psi. \tag{17}$$

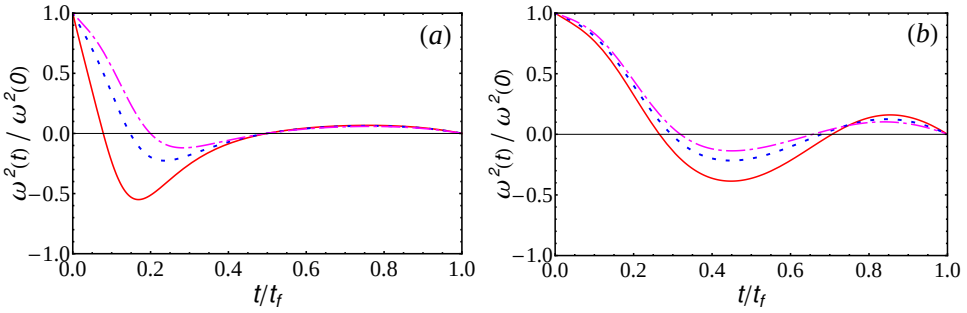

Figure 1. (color online). The squared frequency $\omega^2(t)$ for an expansion from $\omega_0 = 250\times 2\pi$ Hz to $\omega_f = 2.5\times 2\pi$ Hz in $t_f = 6$ ms (a) Polynomial form $b = \sum_{j=0}^{5} a_j t^j$; (b) Exponential of a polynomial $b = \exp\sum_{j=0}^{5} c_j t^j$. In both figures: 1D, TF (solid, red line); 2D, or 1D with $g(t) = g_0/b(t)$, or 3D with $g(t) = g_0 b(t)$ (dotted, blue line); 3D, TF (dot-dashed, magenta line).

This is then the ideal situation for designing a frictionless process by shaping b and ω exactly as in the 1D Schrödinger equation, i.e., using (9) and (10).

Finally, the case $d = 3$ is considered. It is somewhat similar to 1D, in the sense that the generic case leads to time-dependent coefficients in the auxiliary equation. Similarly to 1D, by using Eqs. (5) and (6) the time-independence of the coefficients in the auxiliary equation would require now a time dependent coupling of the form $g(t) = g_0 b(t)$; alternatively, in the Thomas-Fermi regime and with g constant, all coefficients become time independent with

$$\ddot{b} + \omega(t)^2 b = \frac{\omega_0^2}{b^4}, \quad \tau(t) = \int_0^t \frac{dt'}{b^3}, \tag{18}$$

and in this case the boundary condition for $b(t_f)$ in (10) should be modified to $b(t_f) = (\omega_0/\omega_f)^{2/5}$, assuming again $\ddot{b}(0) = \ddot{b}(t_f) = 0$.

4. Examples

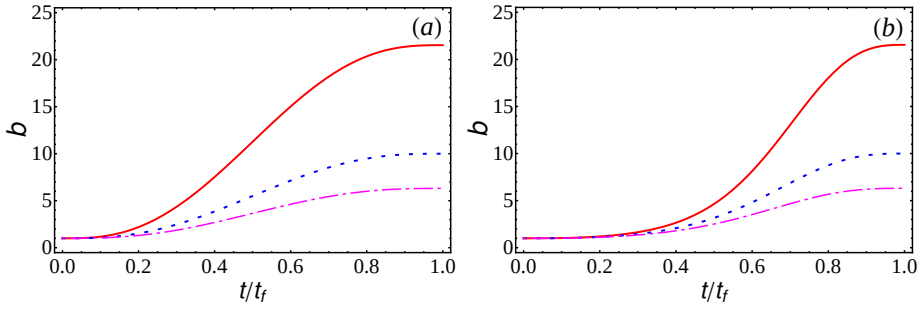

Figure 2. (color online). b corresponding to Fig. 1, $b(t_f) = (\omega_0/\omega_f)^{2/\nu}$, with $\nu = 3$ (solid, red), $\nu = 4$ (dotted, blue), and $\nu = 5$ (dot-dashed, magenta). (a) Polynomial b, (b) b is the exponential of a polynomial.

Let us consider an expansion reducing the frequency 100 times from $250 \times 2\pi$ Hz to $2.50 \times 2\pi$ Hz in 6 ms. This time is too short for the condensate to follow any $\omega(t)$ adiabatically at all times [5] but with our designed trajectories and thanks to the expulsive interval which accelerates the spreading, the final state would indeed be the same, up to a global phase, than the state obtained if a slow process could be implemented for such an expansion. For the regular Ermakov equation (5), a polynomial $b(t)$, and $\omega_0 \gg \omega_f$, a simple estimate is that a repulsive time interval is necessary for $t_f < 1/(2\omega_f)$. Figure 1 shows frequency trajectories for the different cases discussed before and two different ansatz of b. Higher powers of b in the right hand side of the Ermakov-like equations (corresponding to higher dimensions in the TF regime) imply a smaller increment for b during the expansion, see Fig. 2, which makes the change of ω smoother as well. It is remarkable that for a fixed g (for 2D, or the TF regimes in 1D and 3D), the non-linearity does not play any role in the design of optimal (frictionless) frequency trajectories. They only depend on the initial and final frequencies, the available time t_f and the functional form chosen for $b(t)$.

5. Discussion

We will provide here some complementary details and relate the results to other works. An important remark on the TF approximation used for 1D and 3D geometries is that the non-linear coupling cannot be arbitrarily strong. The condition $g|\Psi|^2/(\hbar\omega_0) \gg 1$ should be compatible with the derivation of the 1D GP equation [10] in a weak interaction limit, i.e., $a_s|\psi|^2 << 1$, where a_s is the s-wave scattering length.

For completeness we should mention an alternative to the steps given after Eq. (4). We may as well impose that the coefficient multiplying $\rho^2 b^4 \Psi$ must vanish instead of becoming a non-zero constant [13, 14, 15]. This amounts to imposing $\ddot{b} + \omega(t)^2 b = 0$ instead of the Ermakov equation (5). Proceeding as in Sec. 2 with τ given by Eq. (6), the resulting auxiliary equation becomes

$$i\hbar\frac{\partial\Psi}{\partial\tau} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial\rho^2} + gb|\Psi|^2\Psi, \tag{19}$$

which is not an equation for the harmonic oscillator but for a condensate without confining external fields and with a, generically, time dependent non-linear coupling factor. Adapting the time dependence of g as $g(t) = g_0/b(t)$, this method provides, from known analytical solutions of Eq. (19) with a constant factor $g(t)b(t) = g_0$, explicit solutions that have been used in the context of soliton dynamics [13, 14, 15]. While the solutions $\psi(x,t)$ for the same $\omega(t)$ and initial conditions should of course be equivalent to the ones obtained with the ordinary Ermakov equation, we find the later better suited for the application of our inverse technique.

In summary, it is possible to take a Bose-Einstein condensate in a very short time from an initial harmonic trap to a final one without excitations, by choosing the time dependence of the frequency according to the Ermakov equation or its modifications after matching the time dependence of a scaling factor to suitable boundary conditions.

In 1D and 3D traps this requires either a simultaneous change of the time-dependence of the coupling, or a Thomas-Fermi type of regime. 2D traps are privileged in this respect and do not require either of these conditions. Their peculiar symmetry properties were already noticed by Pitaevskii and Rosch [16]. Indeed, the 2D geometry allows for an extension of the present results beyond the GP equation framework by expanding perturbatively the field operator around the condensate wavefunction, and treating the perturbation with an ansatz parallel to (15) and the same phase [2].

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